What Is Described: Geometry, Energy, and Time in Constitutive Modeling

Prologue

Constitutive modeling is often taught as a catalogue.

Hooke’s law for elasticity. Newton’s law for viscosity. Yield surfaces for plasticity. Damage variables for fracture. Relaxation spectra for polymers. Hyperelastic potentials for rubber. Flow rules for metals. Internal variables for memory, hardening, damage, or relaxation.

Each appears to be a separate invention, designed for a separate material behavior.

But beneath this catalogue lies a deeper structure.

Almost every constitutive model, from the simplest elastic law to a sophisticated viscoelastic-plastic-damage framework, can be read as an answer to three questions:

What is the material state we have chosen to describe?

What energy is stored in that state?

How does that state evolve?

In other words, constitutive modeling is organized around three ideas:

Geometry. Energy. Time.

Here, geometry does not mean physical shape alone. It means the chosen structure of material state space: the variables, configurations, fields, constraints, internal coordinates, and admissible transformations through which a material state is represented.

Energy gives that state mechanical meaning. It tells us what the material stores, what it resists, what it releases, and what driving forces are generated.

Time is the evolution of the material state through the chosen state space. Because this evolution is physical, not merely mathematical, it is constrained by the second law: admissible processes must respect the direction of non-negative dissipation.

In materials with memory, viscosity, damage, diffusion, aging, or rate dependence, the evolution law may also introduce intrinsic time scales. These time scales are not abstract clocks. They are meaningful only relative to the material mechanism and to the state variables chosen to represent it.

Thus:

Geometry defines what the material state can be.

Energy defines what can be stored, released, and used to drive change.

Time is the thermodynamically admissible evolution of that state.

This distinction is important because the three ideas are not equal.

Energy is constrained by conservation.

Time is constrained by dissipation.

Geometry is different. Geometry is selected.

This asymmetry is easy to miss. Yet it may be one of the deepest truths in constitutive modeling. It also points toward something broader: in mechanics and physics, progress often begins when the right geometry of description is found.

Energy is conserved.
Time is directed.
Geometry is selected.

And the quality of a constitutive model depends, before anything else, on whether that selected geometry is the right one for the physical question being asked.

I. The Triad

Every constitutive model begins by choosing a state.

That choice is geometric in the broadest sense.

A material model does not describe “the material” in its total physical reality. It describes a selected state of the material. That state may be simple or rich, observable or partly hidden, purely kinematic or enlarged by internal variables.

In schematic form, we may write:

$$z(t) \in \mathcal{S}$$

where $z(t)$ is the material state at time t, and $\mathcal{S}$ is the chosen state space.

The geometry is $\mathcal{S}$: the structure of possible material states.

Time is the evolution of $z(t)$ through that space.

But this evolution is not just a mathematical curve. In constitutive modeling, a change of state is a physical process. If the process is irreversible, dissipative, history-dependent, or rate-dependent, then the second law and the material evolution law become inseparable from the meaning of time.

For small-strain elasticity, the state may be described by the infinitesimal strain tensor. For finite-strain elasticity, it may be described by the deformation gradient, stretch tensors, invariants, or principal stretches. For purely elastic behavior, the response has no intrinsic material time scale; the state is determined by the current deformation and stored energy.

For plasticity, the state may be enlarged by plastic deformation, hardening variables, or an intermediate configuration. In rate-independent plasticity, physical clock time may largely disappear from the constitutive law, but process direction and loading history remain essential. In viscoplasticity, time re-enters through rate sensitivity, viscosity, activation, or mobility.

For damage, the state may include scalar, tensorial, or field-like measures of loss of integrity. Time may appear as real time, cycle count, accumulated exposure, temperature-assisted kinetics, or energy-driven degradation.

For viscoelasticity, the state may contain internal variables carrying fading memory and relaxation. Here intrinsic time scales are central. The present deformation alone is insufficient because the material remembers part of its past.

For complex fluids, the state may involve velocity fields, conformation tensors, orientation distributions, transported internal structures, diffusion, relaxation, and objective rates. In such models, time is inseparable from transport and evolution.

In each case, the model begins by answering:

What state of the material is worth describing?

This is not a trivial decision. The strain tensor, the deformation gradient, the plastic intermediate configuration, the damage variable, the relaxation tensor, the orientation distribution, the crack phase field —none of these is merely a symbol. Each one defines what kind of material world the model is allowed to see.

In solid mechanics, this geometry is often present without being explicitly named. It appears under the language of kinematics, configurations, strain measures, objectivity, compatibility, material symmetry, constraints, and state variables. But taken together, these define the geometry of the constitutive description: the structure within which a material state can be represented, distinguished, transformed, stored, and evolved.

Traditional teaching can obscure this point.

Compatibility, for example, is usually introduced as a purely kinematic condition: a strain field must be compatible with a continuous displacement field. That statement is correct, but it can make compatibility appear secondary, as if it were only a mathematical condition placed before mechanics begins.

In a deeper sense, compatibility is already part of the geometry of admissible material states. It defines what kind of deformation field is allowed within the chosen continuum description. When compatibility is enforced, the model describes a body whose deformation can be represented by a continuous motion. When controlled incompatibility is introduced—through plasticity, defects, growth, damage, or other internal structures—the geometry of the state space has changed.

But compatibility is not the whole of geometry.

This is important.

Plasticity and defect mechanics may require incompatibility. Damage may require a representation of loss of integrity. Viscoelasticity may require memory rather than incompatibility. Fluids may require transport, vorticity, objective rates, or conformation fields rather than strain compatibility in the classical solid-mechanics sense.

The common point is not that all these models share the same mechanism.

The common point is that each model chooses a state space.

Once the state is chosen, energy is assigned to it.

The free energy, often denoted Ψ, tells us what the material stores, what it resists, and what it prefers. It encodes stiffness, anisotropy, thermal dependence, recoverable deformation, equilibrium response, non-equilibrium response, and sometimes degradation.

Stress emerges from energy. Thermodynamic driving forces emerge from energy. Material preference becomes mathematical.

But energy alone is not enough.

Materials do not merely store energy. They also evolve.

Plastic strain accumulates. Damage grows. Viscous branches relax. Polymer networks remember and forget. Microstructures rearrange. Internal variables move through their own histories.

This requires evolution laws.

Time enters as the evolution of state. The second law gives that evolution its admissible direction. When the material mechanism has its own rate, memory, relaxation, diffusion, aging, or degradation process, the evolution law also introduces the relevant intrinsic time scale.

Thus the basic architecture of a constitutive model is:

A **state space**: the geometry of what is being modeled.

A **free energy**: the stored energetic structure defined on that state space.

An **evolution law**: the rule by which the material moves through that state space in a thermodynamically admissible way.

Different material models differ in their details. But they rarely escape this triad.

II. The Laws That Govern

Energy and time are not free inventions of the modeler.

They are disciplined by thermodynamics.

The first law requires energy balance. Mechanical work, heat flow, stored energy, kinetic energy, and dissipated energy must fit within a consistent accounting. Energy may be transformed, but it cannot be created from nothing or vanish without consequence.

The second law imposes direction. It requires that real material processes be thermodynamically admissible. In continuum mechanics, this usually appears through a dissipation inequality, often expressed locally through the Clausius-Duhem inequality.

In schematic form, the requirement is simple:

$$\mathcal{D} \geq 0$$

where $\mathcal{D}$ represents dissipation.

But the simplicity is deceptive.

This inequality constrains stress definitions, internal variables, flow rules, damage laws, relaxation equations, and coupling terms. It tells us that an evolution law cannot be chosen merely because it fits data or produces convenient numerical behavior. It must also respect the direction of physical admissibility.

The second law gives time its thermodynamic direction.

But the second law does not, by itself, determine every material evolution law.

It does not tell us whether a polymer branch relaxes in milliseconds or hours. It does not tell us whether damage accumulates over one cycle or one million cycles. It does not tell us whether creep is negligible or dominant. It does not tell us how temperature shifts the effective time scale of a material process.

Those are material facts.

They must be modeled, measured, calibrated, and interpreted.

A hyperelastic model satisfies the thermodynamic structure by deriving stress from stored energy and requiring no intrinsic material time scale.

A viscoelastic model must ensure that relaxation dissipates energy rather than creates it, while also representing the relevant relaxation spectrum.

A plasticity model must make irreversible flow consistent with non-negative plastic dissipation. If it is rate-independent, time enters primarily through loading path and irreversibility. If it is viscoplastic, time enters through rate sensitivity and flow kinetics.

A damage model must ensure that degradation consumes or releases stored energy in an admissible way, while also representing the relevant kinetics of failure, exposure, fatigue, or aging.

Thermodynamics does not tell us exactly which constitutive model to use. It does not choose the correct damage variable, the correct relaxation spectrum, the correct yield surface, the correct strain-energy density, or the correct internal state.

But it judges every proposed model.

The first law says: the energy accounting must balance.

The second law says: the evolution must have an admissible direction.

The constitutive evolution law says: this is how the chosen state changes.

These are not optional additions to constitutive modeling. They are the frame within which constitutive modeling becomes physics rather than curve fitting.

III. The Asymmetry of Geometry

Geometry is different.

It is not unconstrained. A geometric description must respect objectivity, material symmetry, kinematic compatibility or controlled incompatibility, dimensional consistency, scale, and experimental observability. It must be appropriate to the deformation mode and the mechanism being represented.

But there is no universal law that tells the modeler, once and for all, what the correct geometry of description must be.

This is the fundamental asymmetry:

Energy is conserved.
Time is directed.
Geometry is selected.

The first law governs energy.

The second law governs admissible evolution.

But geometry is what these laws act upon.

If we choose infinitesimal strain as the state, the laws operate on that description.

If we choose the deformation gradient, the laws operate there.

If we introduce a damage variable, the laws constrain its energetic role and its evolution.

If we introduce a plastic intermediate configuration, the laws judge whether the resulting stress and flow are admissible.

If we introduce viscoelastic internal variables, the laws judge whether their relaxation dissipates energy and represents memory consistently.

But the initial act—the decision about what kind of state space is worth describing—is a modeling decision.

That decision is not arbitrary. It is constrained by physics, experiment, scale, mechanism, and purpose. But it is not uniquely dictated by thermodynamics.

This is where constitutive modeling becomes both scientific and creative.

A poor geometric choice can make a model formally elegant but physically weak.

A good geometric choice can make a difficult material behavior suddenly intelligible.

This is why constitutive modeling is not simply the fitting of stress-strain curves. It is the construction of a meaningful material state space.

The deepest question is not always:

Which equation fits the data?

Very often, the deeper question is:

What is the model actually describing?

IV. Internal Variables: Enlarging the State Space

The power of modern constitutive modeling lies largely in the internal variable framework.

Internal variables allow a continuum model to represent mechanisms that are not directly visible at the continuum scale.

But they should not all be interpreted in the same way.

In plasticity, internal variables may carry irreversible rearrangement, residual deformation, hardening, or controlled incompatibility.

In damage, they may carry the loss of stiffness, integrity, or load-carrying capacity.

In viscoelasticity, they may carry fading memory and relaxation.

In polymeric fluids, they may carry molecular conformation, orientation, or stretch.

The common point is not incompatibility.

The common point is that the observable deformation alone is not a sufficient state description.

In this sense, an internal variable is not merely an added parameter. It is an added coordinate in the material state space. It enlarges the geometry of description so that the continuum can represent something the observable deformation alone cannot carry: memory, irreversibility, damage, hardening, relaxation, hidden microstructural change, or loss of integrity.

For elastic behavior, one might describe the state simply as

$$z = \mathbf{F}.$$

For damage, the state may become

$$z = (\mathbf{F}, d),$$

where d is not literally a crack, but an internal coordinate representing loss of material integrity.

For plasticity, the state may become

$$z = (\mathbf{F}, \mathbf{F}^p, \alpha),$$

where $\mathbf{F}^p$ and α carry irreversible deformation and hardening.

For viscoelasticity, the state may become

$$z = (\mathbf{F}, \mathbf{Q}_1, \mathbf{Q}_2, \ldots),$$

where the $\mathbf{Q}_i$ carry memory, relaxation, and non-equilibrium response.

These internal coordinates are not the microscopic mechanisms themselves. They are continuum-level representations of what the macroscopic model must remember.

A scalar damage variable is not literally a crack.

But when placed inside a free energy that degrades stiffness, and when governed by an evolution law driven by an energy release rate, it can represent progressive loss of load-carrying capacity.

A plastic deformation tensor is not literally a dislocation structure.

But through a kinematic decomposition, such as

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p,$$

it allows the model to separate recoverable elastic deformation from irreversible plastic deformation. The total body may remain compatible at the level of observable motion, while the internal geometry carries the memory of permanent rearrangement.

A viscoelastic internal variable is not literally a polymer chain.

But if it evolves with a relaxation time and dissipates energy, it can represent delay, memory, creep, recovery, and hysteresis.

This is the ingenuity of internal variables.

They allow a continuum description to carry traces of mechanisms occurring beneath its own scale of resolution.

The internal variable is a coordinate in the chosen state space.

The free energy gives it mechanical meaning.

The evolution law gives it time.

The dissipation inequality gives it admissibility.

When the mechanism requires one, the material parameters give it an intrinsic time scale.

This is why internal variables are not merely mathematical devices. They are controlled acts of representation.

They say: the continuum cannot see every microscopic event, but it must remember something about them.

The art is deciding what that “something” should be.

V. Model Credibility

A constitutive model is credible only when geometry, energy, and time are aligned.

The geometry must be appropriate to the physical question.

The energy must represent what the material can store and release.

The evolution law must represent the relevant mechanism and its admissible progression.

If any one of these is wrong, the model may still produce numbers. It may even fit a limited set of tests. But it may not answer the intended engineering or scientific question.

A rubber model may fit a uniaxial curve and still fail under multiaxial deformation if its energy form does not preserve the correct deformation architecture.

A damage model may degrade stiffness smoothly and still describe no physically meaningful failure mechanism.

A viscoelastic model may match one frequency range while becoming misleading outside the relevant time scale.

A plasticity model may be numerically convenient while choosing the wrong internal description of irreversibility.

A rate-dependent model may look physically sophisticated while carrying time constants that have no relation to the material process being modeled.

This is why constitutive modeling cannot be reduced to parameter calibration.

Calibration matters. Data matter. Numerical implementation matters. But before calibration, there is representation.

What is the state space?

What is stored?

What evolves?

What is reversible?

What is irreversible?

What is observable?

What is hidden?

What is being homogenized?

What must remain explicit?

Does the mechanism require an intrinsic time scale?

These questions determine whether a constitutive model has physical credibility.

A model with many parameters is not necessarily rich.

A model with few parameters is not necessarily simple-minded.

The real issue is whether the chosen geometry, energy, and time structure match the mechanism being represented.

A good constitutive model does not reproduce every physical detail. It preserves the essential architecture of the material behavior needed for the question at hand.

VI. The Art and the Science

The science of constitutive modeling lies in respecting the laws.

Energy must balance.

Dissipation must be non-negative.

Stress must be objective.

Material symmetry must be respected.

Evolution laws must be thermodynamically admissible.

If intrinsic time scales are introduced, they must be physically meaningful.

These requirements are not matters of taste. They are the discipline that prevents a model from becoming merely a numerical device.

The art lies in choosing the right description.

What state variables matter?

What memory must be retained?

What mechanisms can be homogenized?

What scale is being represented?

What should be hidden?

What must be made explicit?

What geometry is rich enough to answer the question, but not so rich that it becomes unidentifiable?

Does the material need a clock, or only an admissible path?

This is why a constitutive model is never just a formula.

It is a claim about what kind of material world matters for the problem.

The strain-energy function in rubber elasticity is not only a curve fit. It is a claim about which deformation measures carry the essential response.

A yield surface is not only a mathematical boundary. It is a claim about how irreversible deformation begins and evolves.

A damage variable is not only a degradation factor. It is a claim about how loss of integrity can be represented before the model explicitly sees a crack.

A relaxation spectrum is not only a set of times. It is a claim about how a material remembers deformation and how that memory fades.

A master modeler is not someone who knows the largest catalogue of material laws.

A master modeler is someone who knows what needs to be described, what needs to be stored, and how the state must be allowed to evolve.

Epilogue

Geometry, energy, and time.

The described, the conserved, and the directed.

These are not three independent pillars. They are three aspects of one modeling act.

Geometry defines the state space.

Energy gives the state mechanical meaning.

Time is the thermodynamically admissible evolution of the state through that space.

The catalogue of material models is real, but it is not the deepest reality. Beneath the catalogue is a more unified structure.

Every constitutive model asks:

What world have we chosen to describe?

What can be stored in that world?

What can irreversibly change?

What laws must the change obey?

And, when the mechanism requires it, what intrinsic time scale governs that change?

The beauty of constitutive modeling is that it turns material behavior into a disciplined language of state, energy, and evolution.

Its difficulty is that the most important choice—the geometry of description—is also the least automatic.

Energy is conserved.

Time is directed.

Geometry is selected.

And in that selection, constitutive modeling becomes not only mechanics, but judgment.

But this is not only true inside constitutive modeling. The same pattern appears in the larger history of mechanics and physics. Again and again, progress begins when someone finds a new geometry in which nature can be described.

Coda: The Geometry of Discovery

The great moments in mechanics and physics do not always begin with a new equation.

Sometimes they begin with a new geometry.

Mooney saw that rubber could not be understood by forcing it into the language of small-strain elasticity. Large deformation needed its own description: stretches, invariants, and strain-energy functions capable of living in the actual geometry of rubber deformation.

Euler saw that fluids could be described not only by following material particles, but by observing fields at spatial points. Velocity became a field. Motion became something described over space. A new geometry made fluid motion calculable.

Maxwell’s electromagnetism also required a change in description. Electric and magnetic effects were no longer merely forces acting at a distance between charges and currents. They became fields distributed through space and time. Later, in relativistic form, electric and magnetic fields became components of a single electromagnetic tensor. In modern gauge theory, the potential and field strength reveal an even deeper geometric structure: connection, curvature, and invariance.

Einstein saw something still more radical: gravity was not merely a force acting on bodies within a fixed stage. The stage itself had geometry. The metric tensor became the description, and curvature became physics.

In theories of defects, plasticity, and incompatibility, a similar insight appears. A continuum may remain smooth at the level of displacement, while an internal or intermediate configuration carries the trace of irreversible microscopic rearrangement. The defect is not drawn directly as every dislocation or every broken bond. It is represented through a geometry that lets the continuum remember what it cannot explicitly resolve.

In each case, the decisive move was not only analytical.

It was descriptive.

A new geometry made a new physics visible.

Once the geometry was chosen, equations could be written. Energy could be assigned. Balance laws could be enforced. Evolution laws could be constrained. But the first act was the recognition of what kind of state needed to be described.

This is why geometry occupies such a special place.

Energy is governed by conservation.

Time is governed by admissible evolution.

But geometry is the act of selection.

And at the highest level, discovery often begins when someone finds the right geometry in which nature can finally be described.

Before equations can govern a world, someone must first choose the geometry in which that world can exist.